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Approximate Solutions of the Effective Mass Klein-Gordon Equation for Unequal Potentials

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Abstract

Using a suitable approximation scheme to the centrifugal barrier, we solved the 3-dimensional Klein-Gordon equation for effective mass potential under unequal scalar and vector Coulomb-Hulthen potential in the framework of parametric Nikiforov-Uvarov method. The effects of the screening parameter, the effective masses and the potential strengths on energy were graphically and numerically studied in details. It is noted that the relativistic energy of the Klein-Gordon equation under unequal scalar and vector Coulomb-Hulthen potential is highly bounded.

Keywords: Klein-Gordon equation, Eigensolution, Wave equation, Parametric Nikiforov-Uvarov method.

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1. Introduction

The study of exponential-type potentials has drawn much attention of many authors in the relativistic quantum mechanics [1-7] as a result; different authors have devoted interest in the investigation of the analytical solutions of the relativistic wave equations such as the Dirac equation and Klein-Gordon equation. Thus, there are various reports on the bound state solutions of the Klein-Gordon equation with different exponential type potentials of interest such as Hulthen potential [8], Manning-Rosen potential [9], Eckart potential [10] and others. The Klein-Gordon equation has been solved with various traditional techniques like Nikiforov-Uvarov method [11], asymptotic iteration method [12], supersymmetric quantum mechanics [13], Formula method for bound state problem [14], Factorization method [15] and so on. It is noted that in most of the reports, the authors have only obtained the solutions of the Klein-Gordon

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equation with equal vector and scalar potentials. The main objective of this development is to obtain the energy eigenvalue equation which is either bound state or scattering states and the corresponding eigenfunction. Motivated by the interest in the exponential-type potential and the relativistic spinless particle, the authors intend to study the Klein-Gordon equation with a combination of Coulomb and Hulthen potentials with unequal vector and scalar potentials which has not been reported yet. The Hulthen potential is one of the important short-range potentials in physics and it has been applied to a number of areas such as nuclear and particle physics, atomic physics, condensed matter and chemical physics [16, 17]. On the other hand, the knowledge of Coulomb function has played a significant role in the understanding of atomic spectra as well as the electron-ion collision. This however, is seen to be the backbone of quantum defect theory that gives a systematic understanding of atomic spectra near thresholds and other properties of bound and quasi-bound states [18-20]. Thus, the importance of both Hulthen potential and Coulomb potential or their combination in science cannot be over emphasized. The combination of Coulomb and Hulthen potentials is given as [21]

$$V(r) = -\frac{C}{r} + \frac{He^{-\delta r}}{1-e^{-\delta r}}, \tag{1}$$

where C and H are potential strength, δ is the screening parameter and r is the internuclear separation.

The scheme of our work is as follows: In the next section, we briefly give the methodology of parametric Nikiforov-Uvarov method. In section 3, we obtain the bound state solutions. In section 4, we discuss our results and finally, the concluding remark is given in section 5.

2. Parametric Nikiforov-Uvarov Method

In this section, we briefly give the parametric Nikiforov-Uvarov method. To use the methodology of parametric Nikiforov-Uvarov method, Tezcan and Sever [22], considered the following differential equation.

$$\left[\frac{d^2}{ds^2} + \frac{c_1 - c_2 s}{s(1-c_3 s)}\frac{d}{ds} + \frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{s^2(1-s)^2}\right]\psi(s) = 0. \tag{2}$$

According to the parametric Nikiforov-Uvarov method, the eigenvalue and eigenfunction respectively are given as [22]

$$c_2 n + n(n-1)c_3 + c_7 + 2c_3 c_8 - (2n+1)\left(c_5 - \sqrt{c_9} - c_3\sqrt{c_8}\right) + 2\sqrt{c_8 c_9} = 0, \tag{3}$$

$$\psi(s) = s^{c_{12}}(1-s)^{-c_{12}-\frac{c_{13}}{c_3}} P_n^{\left(c_{10}-1, \frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3 s). \tag{4}$$

The parametric constants in Eq. (3) and Eq. (4) are obtain as

$$\left.\begin{array}{l} c_4 = \frac{1}{2}(1-c_1), c_5 = \frac{1}{2}(c_2 - 2c_3), c_6 = c_5^2 + \xi_1, \\ c_7 = c_4 c_5 - \xi_2, c_8 = c_4^2 + \xi_3, c_9 = c_6 + c_3 c_7 + c_3^2 c_8 \\ c_{10} = c_1 + 2c_4 + 2\sqrt{c_8}, c_{11} = c_2 - 2c_5 + 2\left(\sqrt{c_9} + c_3\sqrt{c_8}\right), \\ c_{12} = c_4 + \sqrt{c_8}, c_{13} = c_5 - \left(\sqrt{c_9} + c_3\sqrt{c_8}\right) \end{array}\right\}. \tag{5}$$

3. Bound State Solutions

The energy eigenvalue equation and the corresponding wave functions are obtain in this section. The time-independent Klein-Gordon equation with the scalar potential $S(r)$ and vector potential $V(r)$ in the relativistic unit ($\hbar = c = 1$) is given by

$$\left[\frac{d^2}{dr^2} + (M + S(r))^2 - (E_{n\ell} - V(r))^2 - \frac{\ell(\ell+1)}{r^2}\right]R_{n\ell}(r) = 0, \tag{6}$$

where M is mass of particle, $E_{n\ell}$ is the relativistic energy of the system and $R_{n\ell}(r)$ is the wave function. The Klein-Gordon equation given in Eq. (6) is for a potential $2V$ in the non-relativistic limit which can never give the equation results for the Schrdinger equation. However, Alhaidari et al. [22], pointed out that for the equation describing a scalar particle (spin-0 particle), the choice of the potential is $S = +V$ which results into a nontrivial non-relativistic limit. Thus, Eq. (6) becomes

$$\left[\frac{d^2}{dr^2} + \left(M + \frac{1}{2}S(r)\right)^2 - \left(E_{n\ell} - \frac{1}{2}V(r)\right)^2 - \frac{\ell(\ell+1)}{r^2}\right]R_{n\ell}(r) = 0. \tag{7}$$

The purpose of this study is to investigate critically the solution of the Klein-Gordon equation with unequal scalar and vector potentials for an effective mass. Thus the scalar potential and vector potential respectively becomes

$$S(r) = -\frac{s_0}{r} + \frac{s_1 e^{-\delta r}}{1 - e^{-\delta r}}, \tag{8}$$

$$V(r) = -\frac{v_0}{r} + \frac{v_1 e^{-\delta r}}{1 - e^{-\delta r}}, \tag{9}$$

where s_0, $s_1$are first and second strength of the scalar potential and v_0, v_1 are first and second strength of the vector potential. For effective mass Klein-Gordon equation,

$$M = m_0 + \frac{m_1 e^{-\delta r}}{1 - e^{-\delta r}}. \tag{10}$$

A physical examination of Eqs. (7), (8) and (9) revealed that Eq. (7) cannot be solved for $\ell = 0$, therefore, we resort to employ the following approximation scheme to deal with the centrifugal term

$$\frac{1}{r^2} \approx \frac{\delta^2}{\left(1 - e^{-\delta r}\right)^2}. \tag{11}$$

Substituting Eqs. (8), (9), (10) and (11) into Eq. (7) and by defining a variable of the form $y = e^{-\delta r}$, Eq. (7) turns to

$$\left[\frac{d^2}{dy^2} + \frac{1-y}{y(1-y)}\frac{d}{dy} + \frac{-Ay^2 + By - C}{y^2(1-y)^2}\right]R_{n\ell}(r) = 0, \tag{12}$$

where

$$A = \frac{2m_0 m_1 + m_0 s_1 + E_{n\ell} v_1 + E_{n\ell}^2 - m_0^2 - m_1(m_1 + s_1)}{\delta^2} + \frac{-s_1^2 - v_1^2}{4\delta^2}, \tag{13}$$

$$B = \frac{2m_0 m_1 + m_0 s_1 + E_{n\ell} v_1 + 2E_{n\ell}^2 - 2m_0^2}{\delta^2} + \frac{v_0 v_1 - s_0 s_1 - 2m_1 s_0 + 2m_0 s_0 + 2E_{n\ell} v_0}{2\delta} \tag{14}$$

$$C = \frac{v_0^2 - s_0^2}{4} + \frac{E_{n\ell}^2 - m_0^2}{\delta^2} + \frac{m_0 s_0 + E_{n\ell} v_0}{\delta} + \ell(\ell+1) \tag{15}$$

Comparing Eq. (12) with Eq. (2), Eq. (5) becomes

$$\left. c_1 = c_2 = c_3 = 1, c_4 = 0, c_5 = -\frac{1}{2}\right\}. \tag{16}$$

$$c_6 = \frac{1}{4} + \frac{2m_0 m_1 + m_0 s_1 + E_{n\ell} v_1 + E_{n\ell}^2 - m_0^2 - m_1(m_1 + s_1)}{\delta^2} + \frac{-s_1^2 - v_1^2}{4\delta^2}, \tag{17}$$

$$c_7 = -\left(\frac{2m_0 m_1 + m_0 s_1 + E_{n\ell} v_1 + 2E_{n\ell}^2 - 2m_0^2}{\delta^2} + \frac{v_0 v_1 - s_0 s_1 - 2m_1 s_0 + 2m_0 s_0 + 2E_{n\ell} v_0}{2\delta}\right), \tag{18}$$

$$c_8 = \frac{v_0^2 - s_0^2}{4} + \frac{E_{n\ell}^2 - m_0^2}{\delta^2} + \frac{m_0 s_0 + E_{n\ell} v_0}{\delta} + \ell(\ell+1), \tag{19}$$

$$c_9 = \frac{(1+2\ell)^2\delta^2 + v_1^2 - s_1^2 + \delta^2\left(v_0^2 - s_0^2\right) + 2(s_0s_1 - v_0v_1) - 4m_1(m_1 + s_1 - s_0\delta)}{4\delta^2}, \tag{20}$$

$$c_{10} = 1 + 2\sqrt{\frac{v_0^2 - s_0^2}{4} + \frac{E_{n\ell}^2 - m_0^2}{\delta^2} + \frac{m_0s_0 + E_{n\ell}v_0}{\delta} + \ell(\ell+1)}, \tag{21}$$

$$c_{11} = 2\left(1 + \sqrt{C}\right) + \frac{1}{\delta}\sqrt{(1+2\ell)^2\delta^2 + v_1^2 - s_1^2 + \delta^2\left(v_0^2 - s_0^2\right) + 2(s_0s_1 - v_0v_1) - 4m_1(m_1 + s_1 - s_0\delta)}, \tag{22}$$

$$c_{12} = \sqrt{\frac{v_0^2 - s_0^2}{4} + \frac{E_{n\ell}^2 - m_0^2}{\delta^2} + \frac{m_0s_0 + E_{n\ell}v_0}{\delta} + \ell(\ell+1)}, \tag{23}$$

$$c_{13} = -\sqrt{C} - \frac{1}{2\delta}\left(\delta + \sqrt{(1+2\ell)^2\delta^2 + v_1^2 - s_1^2 + \delta^2\left(v_0^2 - s_0^2\right) + 2(s_0s_1 - v_0v_1) - 4m_1(m_1 + s_1 - s_0\delta)}\right). \tag{24}$$

Substituting the values of c_1 to c_9 in Eq. (16) to Eq. (20) into Eq. (3) we have energy equation as

$$\lambda_{T_0} + E_{n\ell}^2 - \delta^2\left[\frac{\lambda_{T_1} + \lambda_{T_2} - \left(n + \frac{1}{2}\right)\sqrt{\lambda_{T_3} + \lambda_{T_4}}}{1 + 2n + \sqrt{\lambda_{T_3} + \lambda_{T_4}}}\right]^2 = 0, \tag{25}$$

and the corresponding wave function is obtain by substituting the values of c_{10} to c_{13} in Eq. (21) to Eq. (24) into Eq. (4):

$$R_{n\ell}(y) = N_{n\ell}\, y^{\sqrt{C}}(1-y)^{0.5+\sqrt{\lambda_{T_3}+4\lambda_{T_4}}}\, P_n^{\left(2\sqrt{C}+\sqrt{\lambda_{T_3}+4\lambda_{T_4}}\right)}(1-2y), \tag{26}$$

where

$$\lambda_{T_0} = \delta^2\left(\frac{v_0^2 - s_0^2}{4}\right) + \delta\left(m_0s_0 + Ev_0\right) + \ell(\ell+1)\delta^2 - m_0^2, \tag{27}$$

$$\lambda_{T_1} = \frac{s_0}{2}\left(s_0 - \frac{s_1}{\delta}\right) + \frac{v_0}{2}\left(\frac{v_1}{\delta} - v_0\right) - \frac{s_0}{\delta}\left(m_0 + m_1\right), \tag{28}$$

$$\lambda_{T_2} = \frac{E_{n\ell}}{\delta}\left(\frac{v_1}{\delta} - v_0\right) + \frac{m_0}{\delta^2}\left(2m_1 + s_1\right) - n\left(n+1\right) - 0.5. \tag{29}$$

$$\lambda_{T_3} = \frac{v_1^2 - s_1^2 + 4m_1(s_0\delta - m_1 - s_1)}{\delta^2} + (1+2\ell)^2, \tag{30}$$

$$\lambda_{T_4} = s_0\left(\frac{2s_1}{\delta} - s_0\right) + v_0\left(v_0 - \frac{2v_1}{\delta}\right), \tag{31}$$

4. Discussion

In our graphs, we make all the plots with positive energies. In Fig 1, we examined the particle's ground state energy of the scalar potential with the particle's mass m_0. It is observed that when the mass m_0 increases, the particle's energy also increases and when the mass decreases, the energy decreases. Thus, as mass m_0 decreases, the particle becomes more bound. In Fig 2, we examined the particle's ground state energy of the vector potential with the particle's mass m_0. It is observed that when the mass m_0 increases, the particle's energy for the vector potential increases. It is noted that the particle becomes bounded before $m_0 = 0$. In Fig 3, we plotted energy of the vector potential against the screening parameter. It is seen that as the screening parameter increase, the energy of the vector potential decreases. However, a further or an extension of the graph will show that at δ >0.7, the particle will be bounded. Thus, this particle interacts with the system significantly in the well for δ >7. In Fig 4, we showed the variation of the energy of

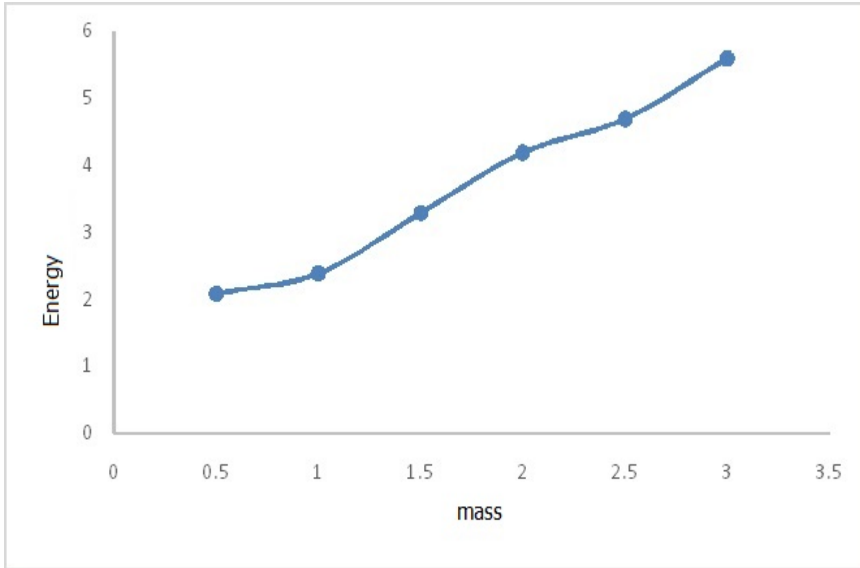

Figure 1. Variation of ground state energy $E_{0,\ell}$ of the scalar potential against the mass m_0 with $v_0 = v_1 = 0$, $s_0 = s_1 = -1$, $\ell = 1$ and $\delta = 0.25$

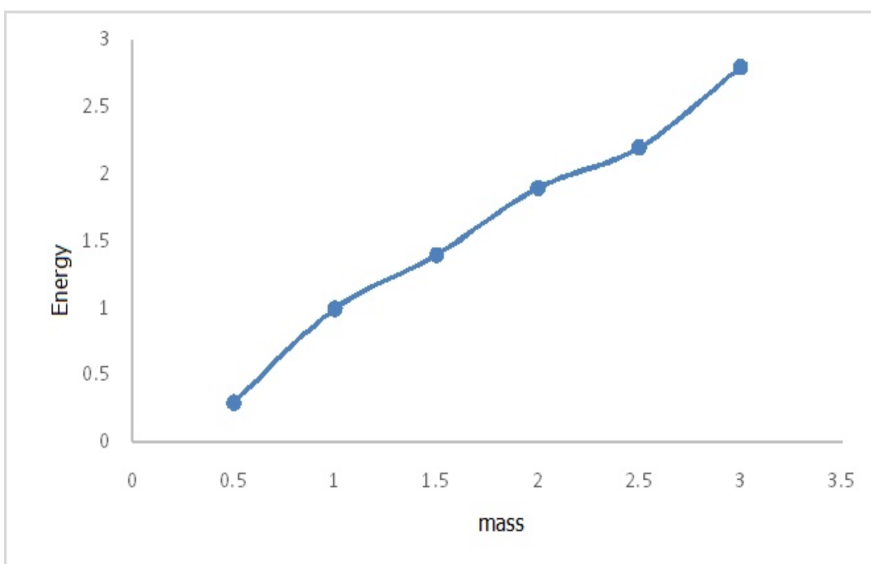

Figure 2. Variation of ground state energy $E_{0,\ell}$ of the vector potential against the mass m_0 with $s_0 = s_1 = 0$, $v_0 = v_1 = -1$, $\ell = 1$ and $\delta = 0.25$

the scalar potential against the screening parameter. This Fig 4 shows that as the screening parameter increases, the energy of the scalar potential for particle decreases. The decrease in the energy of the scalar potential is not as high as that of the vector potential. Thus, it takes particle more time to be bounded for an increase in the screening parameter of the scalar potential. In Figs 5 and 6, we showed the variation of energy against the first and second strengths of the vector potential respectively. In each case, the energy increases as each strength of the potential increases. Figs 7 and 8 respectively, showed the variation of the energy against the first and second strengths of the scalar potential. It is

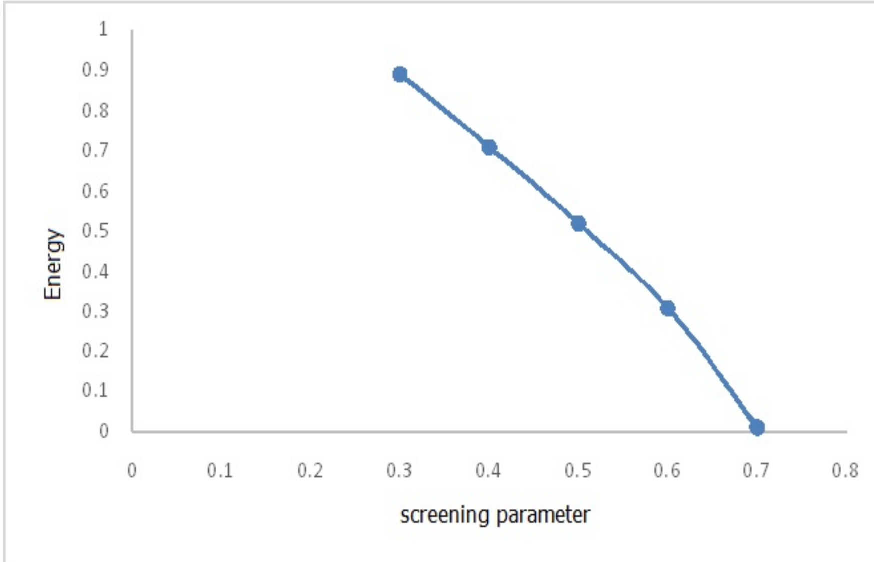

Figure 3. Variation of ground state energy $E_{0,\ell}$ of the vector potential against the screening parameter δ with m_0 = 1, s_0 = s_1 = 0, v_0 = v_1 = -1 and ℓ = 1

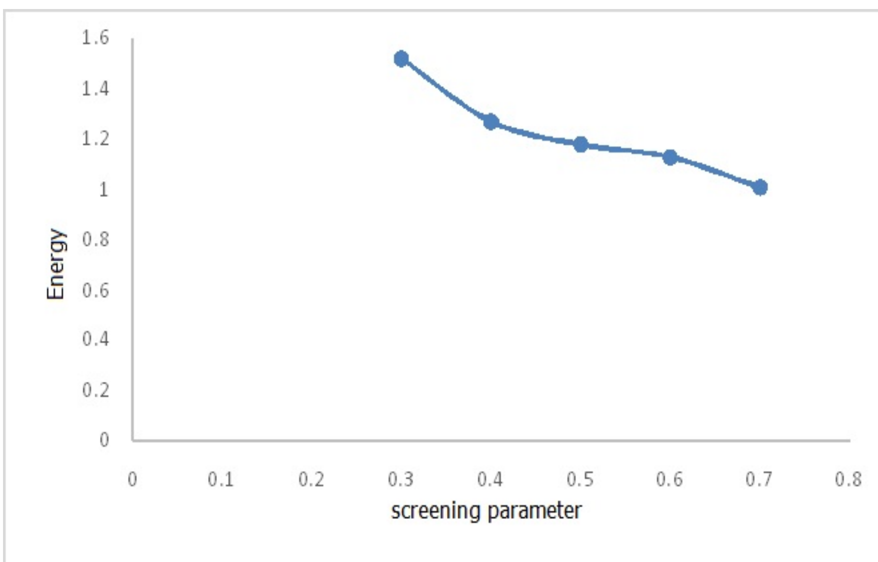

Figure 4. Variation of ground state energy $E_{0,\ell}$ of the scalar potential against the screening parameter δ with m_0 = 1, v_0 = v_1 = 0, s_0 = s_1 = -1 and ℓ = 1

noted in each case that as the strength of the potential increases, the energy of the particle also increases. However, a further decrease in the second strength of the scalar potential beyond 0.6, will makes the particle to be bounded. In our graphs, we make all the plots with positive energies.

In Table 1, we presented energy for various n and ℓ with m_0 = 0.1, m_1 = 0.5 and δ = 0.25. It is seen that the energies obtained for $v_0 = s_0 > v_1 = s_1$ are greater than their counterpart obtained for $v_0 = s_0 < v_1 = s_1$. It is noted

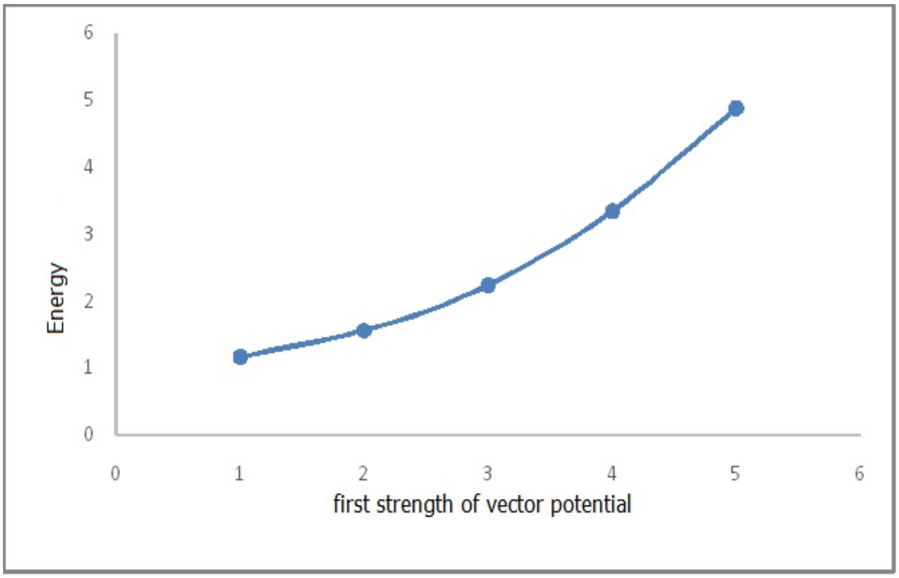

Figure 5. Variation of ground state energy $E_{0,\ell}$ against the first strength of the vector potential with $\delta = 0.25$, $m_0 = 1$, $s_0 = s_1 = v_1 = 0$ and $\ell = 1$

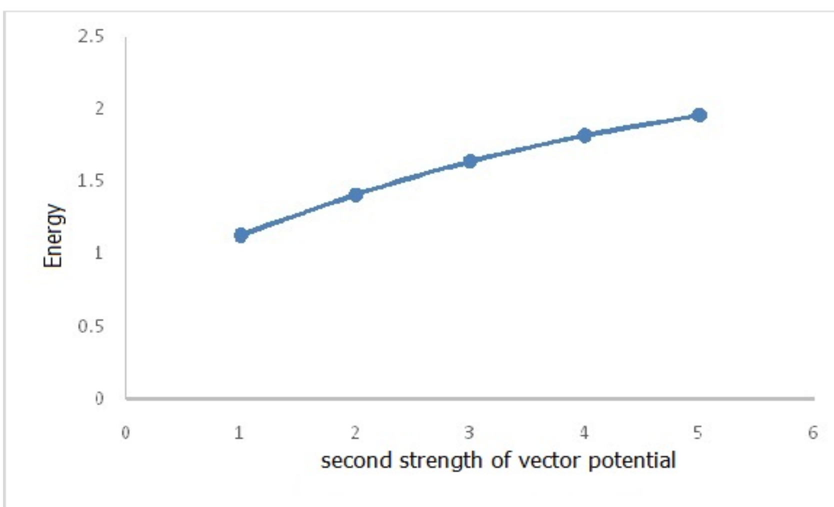

Figure 6. Variation of ground state energy $E_{0,\ell}$ against the second strength of the vector potential with $\delta = 0.25$, $m_0 = 1$, $s_0 = s_1 = v_0 = 0$ and $\ell = 1$

here that the particle is bounded when $v_0 = s_0 < v_1 = s_1$. It is also noted in Table 1 that when $v_0 = s_1$ and $v_1 = s_0$, the particle is highly bounded. However, a change in the magnitude of these parameters has no effect in the energy. In Table 2, we presented energy eigen values for various n and ℓ using the same values of the potential parameters in Table 1 but reversed the numerical values of m_0 and m_1 ($m_0 = 0.5$, $m_1 = 0.1$). It is noted in this case that the particle is less bounded in any case. Thus, the particle is highly bounded when $m_0 < m_1$ and less bounded when $m_1 < m_0$.

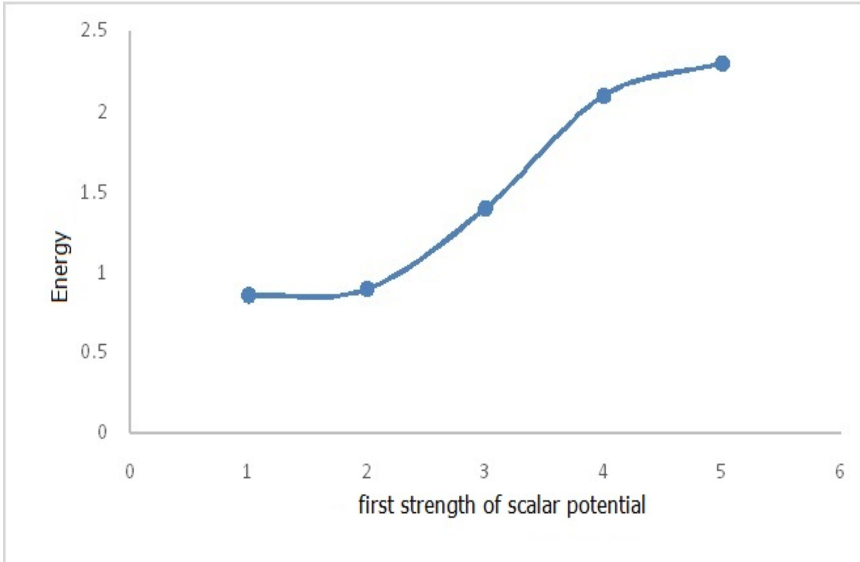

Figure 7. Variation of ground state energy $E_{0,\ell}$ against the first strength of the scalar potential with $\delta = 0.25$, $m_0 = 1$, $s_1 = v_0 = v_1 = 0$ and $\ell = 1$

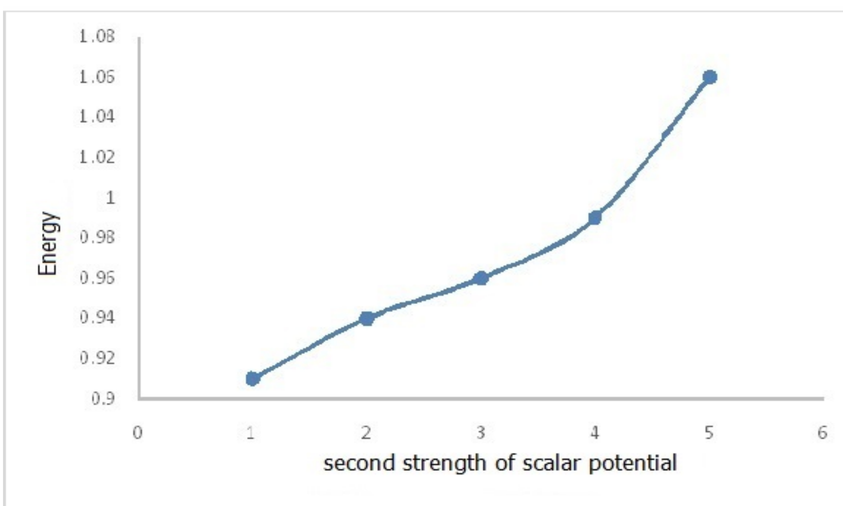

Figure 8. Variation of ground state energy $E_{0,\ell}$ against the second strength of the vector potential with $\delta = 0.25$, $m_0 = 1$, $s_0 = v_0 = v_1 = 0$ and $\ell = 1$

5. Conclusion

In this work, we obtained energy equation and the corresponding wave function of the Klein-Gordon equation. The Coulomb-Hulthen interacting potential which is a combination of two distinct potentials has four potential strengths under unequal scalar (s_0, s_1) and vector (v_0, v_1) potentials. The Klein-Gordon equation was studied under effective mass potential which results to two masses m_0 and m_1. From our results, the energy is more bounded when $v_0 = s_1$,

Table 1. Table 1: Bound State Energy for Various n and ℓ with $\delta = 0.25$, $m_0 = 0.1$ and $m_1 = 0.5$

n	ℓ	$V_0 = S_0 = 2$,	$V_0 = S_0 = -2$,	$V_0 = S_1 = -1$,	$V_0 = S_1 = 2$,
		$V_1 = S_1 = -1$	$V_1 = S_1 = -1$	$V_1 = S_0 = 2$	$V_1 = S_0 = -1$
0	1	2.724997760	-0.054188597	-0.131174682	-0.131174682
		0.088312436	-0.845811403	-2.490401460	-2.490401460
1	1	1.844699907	-0.246875000	0.049960891	0.049960891
		-0.033831848	0.521977909	-1.864352738	-1.864352738
0	2	1.104013400	-0.309375000	-0.836792602	-0.836792602
		0.614993481	0.502328562	0.508443865	0.508443865
1	2	1.212672316	-0.223333333	-0.334407900	-0.334407900
		0.268406430	0.507893252	-1.179669508	-1.179669508
2	2	1.341329611	-0.186458333	-0.034118226	-0.034118226
		0.051882217	0.402594397	-1.426598947	-1.426598947

Table 2. Table 2: Bound State Energy for Various n and ℓ with $\delta = 0.25$, $m_0 = 0.5$ and $m_1 = 0.1$

n	ℓ	$V_0 = S_0 = 2$,	$V_0 = S_0 = -2$,	$V_0 = S_1 = -1$,	$V_0 = S_1 = 2$,
		$V_1 = S_1 = -1$	$V_1 = S_1 = -1$	$V_1 = S_0 = 2$	$V_1 = S_0 = -1$
0	1	-6.284493764	0.562284080	0.075434100	0.147141504
		-0.158036718	-0.945096580	71.43947106	-0.051731917
1	1	5.477280500	0.610290590	0.185331714	0.083333217
		-0.207154024	-0.915485395	-3.772505260	-0.138177226
0	2	6.263655146	0.261765344	-0.215246652	0.446447363
		0.062160934	-0.684043782	-5.183208462	-0.257375185
1	2	2.556178463	0.377712280	-0.380185760	0.326343670
		-0.021030253	-0.722000989	-2.453850088	-0.284565937
2	2	2.040642131	0.502791718	0.123317453	0.208078464
		-0.127382929	-0.812559602	-2.100359461	-0.304401525

$v_1 = s_0$ and $m_0 > m_1$. Finally, the energy obtained, decreases as the screening parameter increases.

References

[1] C. S. Jia, X. I. Zeng & L. T. Sun, "PT Symmetry and Shape Invariance for a Potential Well with a Barrier", Phys. Lett. A **294** (2002) 185.

[2] C. A. Onate, M. C. Onyeaju & A. N. Ikot, "Analytical Solutions of the Dirac Equation under Hellmann-Frost Musulin Potential", Ann. Phys. **375** (2016) 239.

[3] C. A. Onate, "Relativistic and Non-relativistic Solutions of Inversely Quadratic Yukawa Potential", Afr. Rev. Phys. **8** (2013) 325.

[4] A. N. Ikot, E. O. Chukwuocha, M. C. Onyeaju, C. A. Onate, B. I. Ita & M. E. Udoh, "Thermodynamic Properties of Diatomic Molecules with General Molecular Potential", Pramana J. Phys. **90** (2018) 22.

[5] M. C. Onyeaju, A. N. Ikot, C. A. Onate, O. Ebomwonyi, M. E. Udoh & J. O. A. Idiodi, "Approximate Bound-states Solutions of the Dirac Equation with Some Thermodynamic Properties for Deformed Hylleraas Plus Deformed Woods-Saxon Potential", Eur. Phys. J. Plus **132** (2017) 302.

[6] E. Maghsoodi, H. Hassanabadi & Ayedoğdu, "Dirac Particles in the Presence of Yukawa Potential Plus a Tensor Interaction in SUSYQM Framework", Phys. Scr. **86** (2012) 015005.

[7] H. Hassanabadi, S. Zarrinkamar & H. Rahimov, "Approximate Solution of D-Dimensional Klein-Gordon with Hulthen-type Potential via SUSYQM", Commun. Theor. Phys. **56** (2011) 423.

[8] G. Chen, Z. D. Chen & Z. M. Lou, "Exact Bound State Solutions of the S-wave Klein-Gordon Equation with the Generalized Hulthen Potential", Phys. Lett. A **331** (2004) 374.

[9] T. Chen, S. R. Lin & C. S. Jia, "Solutions of the Klein-Gordon Equation with the Improved Rosen-Morse Potential Energy Model", Eur. Phys. J. Plus **128** (2013) 69.

[10] I. O. Akpan, A. D. Antia & A. N. Ikot, "Bound-state Solutions of the Klein-Gordon Equation with q-deformed Equal Scalar and Vector Eckart Potential using a New Improved Approximation Scheme", High Energy Phys (2012).

[11] C. S. Jia, P. Guo & X. L. Peng, “Exact Solution of the Dirac-Eckart Problem with Spin and Pseudospin Symmetry”, J. Phys. A: Math. Gen. **39** (2006) 7737.

[12] A. F. Nikiforov & V. B. Uvarov, “Special Functions of Mathematical Physics”, Birkhauser, Basel (1988).

[13] O. Bayrak & I. Boztosun, “Bound-state Solutions of the Hulthen Potential by using the Asymptotic Iteration Method”, J. Phys. A: Math. Gen. **70** (2007) 92.

[14] F. Cooper, A. Khare & U. Sukhtme, “Supersymmetry Quantum Mechanics”, Phys. Rep. **251** (1995) 267.

[15] B. J. Falaye, S. M. Ikdair & M. Hamzavi, “Formula Method for Bound-state Problems”, Few-Body Syst. **56** (2015) 63.

[16] H. S. Dong, “Factorization Method in Quantum Mechanics”, Springer, the Netherlands (2007).

[17] L. Hulthen, “The Hulthen Potential”, Ark. Mat. AstronFys. A **28** (1942) 5.

[18] Y. P. Varshni, “Eigenenergies and Oscillator Strengths for the Hulthen Potential”, Phys. Rev. A. **41** (1990) 4682.

[19] M. J. Seaton, “Quantum Defect Theory”, Rep. Prog. Phys. **46** (1983) 167.

[20] C. H. Greene, A. R. P. Rau & U. Fano, “General Form of the Quantum Defect Theory”, Phys. Rev. A. **26** (1982) 2441.

[21] C. A. Onate, “Approximate Solutions of the Non-Relativistic Schrodinger Equation with An Interaction of Coulomb and Hulthen Potentials”, SOP Tran. Theor. Phys. **1** (2014) 118.

[22] C. Tezcan & R. Sever, “A General Approach for the Exact Solution of the Schrodinger Equation”, Int. J. Theor. Phys. **48** (2009) 337.

[23] A. Alhaidari, H. Bahlouli & A. Al-Hassan, “Dirac and Klein-Gordon Equations with Equal Scalar and Vector Potentials”, Phys. Lett. A **349** (2006) 87.